Directivity Index and Multi-element Arrays

At the beginning of the last section, we began discussing how it would be possible to increase the response and the signal-to-noise ratio by increasing the number of elements that we used to receive sound. This also led to the formulation of the beam pattern function and drawing the response patterns for a simple two-element array.

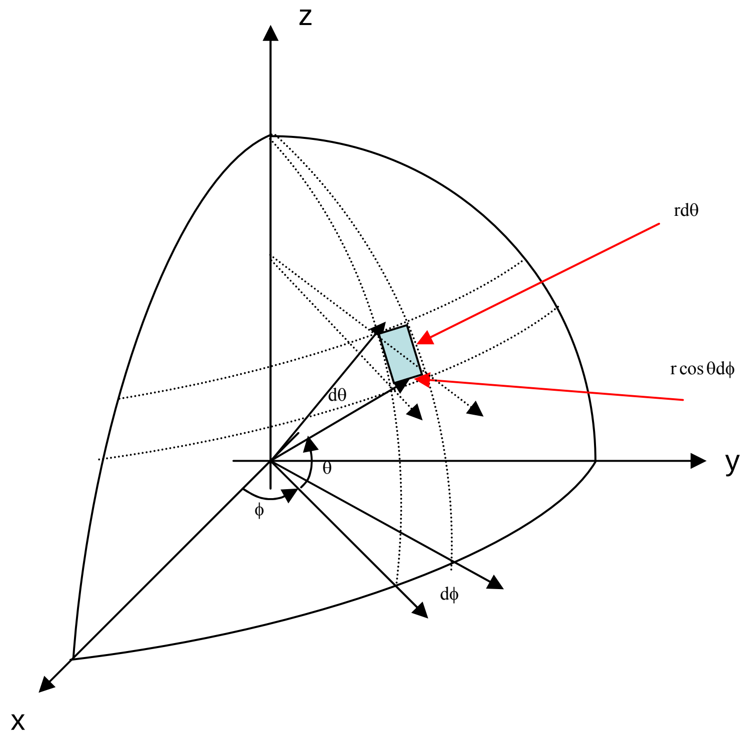
We will quantify the affect of increasing the number of elements in our array by deriving an expression called the Directivity Index. The Directivity Index is the ratio of the total noise power in an isotropic noise filled environment, incident on an array, compared to the power actually received by the system.

$$DI = 10\log\frac{N_{\text{omni-directional noise}}}{N_{\text{directional noise}}}$$

where $N_{\text{omni-directional noise}}$ (N_{ND}) is the power of the isotropic noise incident on the array and $N_{\text{directional noise}}$ (N_D) is the power of the isotropic noise received by the array.

To calculate the Directivity Index of an array,

$$N_{ND} = 4\pi r^2 I_i$$
$$N_D = \iint I_i b(\theta,\phi)\, dA$$



As shown in the above sketch, θ is the latitude angle measured up from the plane of the equator (x-y plane) and ϕ is the longitude angle measured from the x-z axis. The area of a small elemental area on this surface can be found from the following equation, obtained by multiplying the dimensions of the element.

$$dA = r^2 \cos\theta d\theta d\phi$$

The integrations over θ must be from 0 to 2π and the integration over ϕ is from –π/2 to +π/2. When calculating the omni or non-directional power, b =1 and it is easy to show that the integration over θ and ϕ result in a factor of 4π. Similarly, to calculate the directional noise level:

$$N_D = \iint I_i b(\theta,\phi) r^2 \cos\theta d\theta d\phi$$

$$N_D = I_i r^2 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta,\phi) \cos\theta d\theta d\phi$$

Since the beam pattern function is independent of θ such that b(θ,ϕ)=b(θ) and because the beam pattern function is symmetrical about the x-axis, the double integrals can be evaluated as below.

$$N_D = I_i r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta) \cos\theta d\theta \int_0^{2\pi} d\phi$$

$$N_D = 2\pi I_i r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta) \cos\theta d\theta$$

$$N_D = 4\pi I_i r^2 \int_0^{\pi/2} b(\theta) \cos\theta d\theta$$

When this is all combined to calculate the Directivity Index:

$$DI = 10\log\frac{N_{ND}}{N_D}$$

$$DI = 10\log\frac{4\pi r^2 I_i}{4\pi I_i r^2 \int_0^{\pi/2} b(\theta)\cos\theta d\theta}$$

$$DI = 10\log\frac{1}{\int_0^{\pi/2} b(\theta)\cos\theta d\theta}$$

If we can solve the integral of the beam pattern function in the formula above, we can determine the Directivity Index of a given array. The key will be to determine the beam pattern function for the specific array and to evaluate the integral.

Directivity Index for a 2-element Array

If we evaluate the integral in the equation above for a 2-element array, we get the following:

$$DI = 10\log\frac{2}{\left[1+\frac{\sin\left(2\pi d/\lambda\right)}{2\pi d/\lambda}\right]}$$

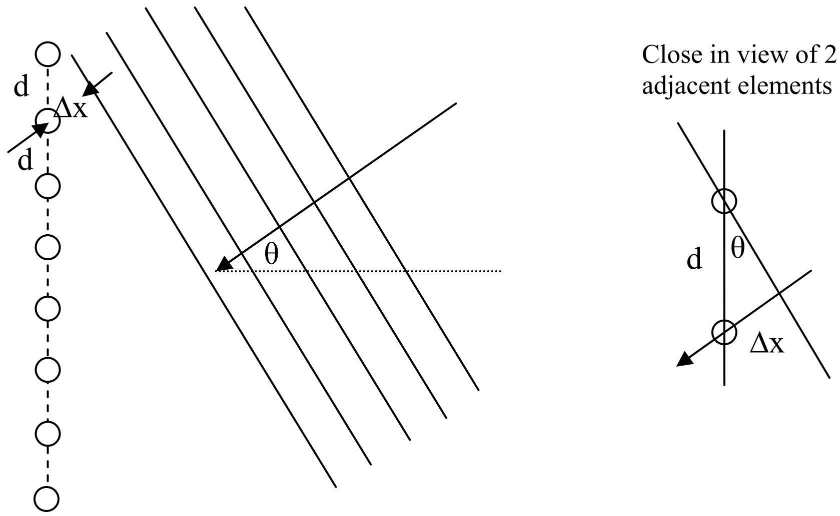
Since the denominator inside the logarithm is simply:

$$\int_0^{\pi/2} b(\theta)\cos\theta d\theta = \int_0^{\pi/2}\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)\cos\theta d\theta = 1+\frac{\sin\left(2\pi d/\lambda\right)}{2\pi d/\lambda}$$

The student should note then that **the Directivity Index of an array varies as a function of frequency** (or wavelength) of the incident sound. When we are evaluating the Directivity Index for an array, normally we will calculate the DI using the center frequency of the frequency band of the processor.

n-Element Array

Beam Pattern Function



We will study an n-element array with separation, d, between elements and an acoustic wave incident at an angle θ just as we did for the two element array. To find the total voltage from all n-elements we have to add up the voltage from each element and then square the result. For the two element case we were able to accomplish this mathematical task using trigonometric identities. The task is more complicated with 3 or more elements so we will use a technique borrowed from electrical engineering called phasor addition.

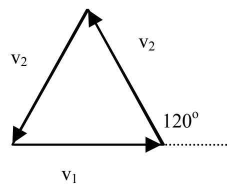
Recall from our electrical engineering that we often used phasor addition to add up AC sin waves in three phase systems. In this technique, the voltage from each array element is represented by a vector-like arrow whose direction is defined by the difference in phase that the element has from the voltage of the adjacent array elements. This "phase angle" representation is where the technique gets its name. The so called "phasor" diagram if formed by connecting the individual "phasors" head to tail analogous to vector addition. If the output from a hypothetical array with three elements each differed by 120^o or $2\pi/3$ radians, the below expressions would represent the output from each:

$$v_1 = V_o \cos(\omega t)$$

$$v_2 = V_o \cos\left(\omega t + \frac{2\pi}{3}\right)$$

$$v_3 = V_o \cos\left(\omega t + \frac{4\pi}{3}\right)$$

If we added up these three voltages, the phasor diagram would appear as below at the time, t=0 sec. If somehow we had an output equal to the sum of these three voltages, the output must be zero volts.

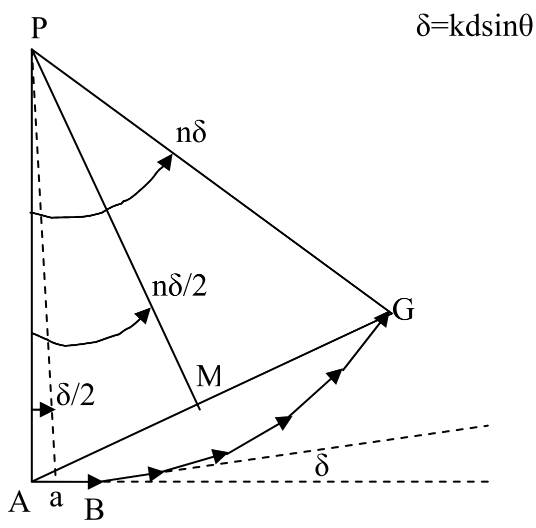


More often in our EE class, we were interested in the difference between 2 phases of a system. We employed phasor subtraction to find the real and reactive parts of this difference. Hopefully it is obvious that the magnitude of the difference between v_1 and v_2 in our example is V_o.

For our multi-element array, the difference in phase between adjacent elements is $\delta = k\,\Delta x$. In the above diagram, we see that each element of the array sees the same wavefront after it has traveled an additional distance $\Delta x = d\sin\theta$ from the element next to it. The phase difference between elements is then $\delta = kd\sin\theta$. The total voltage of beamformer obtained by summing the individual elements is therefore:

$$v_{TOT} = Mp_0\left\{\cos(k[0]+\omega t)+\cos(k[-d\sin\theta]+\omega t)+\cos(k[-2d\sin\theta]+\omega t)+\ldots\cos(k[-(n-1)d\sin\theta]+\omega t)\right\}$$
$$v_{TOT} = v_0\left\{\cos(\omega t)+\cos(\omega t-\delta)+\cos(\omega t-2\delta)+\ldots+\cos(\omega t-(n-1)\delta)\right\}$$
$$v_{TOT} = A\cos(\omega t+\phi)$$

Using a phasor representation, we want to find the resulting amplitude of the sum, A, and sometimes even the resulting phase angle, ϕ. A geometric construction of each of the phasor elements in the sum is drawn as in the diagram below. In this case a 6 element array is shown.



Segment AG is the resulting amplitude of the sum, A. We see that the phasors are approximating the arc of a circular path of radius, R, such that

$$\sin\frac{\delta}{2} \approx \frac{V_0/2}{R} \text{ where R is distance AP}$$

since $\angle$ APa is $\frac{\delta}{2}$. Similarly, since $\angle$ APM = $\frac{n\delta}{2}$ and the midpoint of the chord is $\frac{A}{2}=\frac{V(\theta)}{2}$

$$\sin\left(\frac{n\delta}{2}\right)\approx\frac{V(\theta)/2}{R} \quad \text{Where R is distance AP}$$

Combining these two results and solving for V(θ),

$$V(\theta)=nV_0\left[\frac{\sin n\frac{\delta}{2}}{n\sin\frac{\delta}{2}}\right]=A$$

It is customary to write nV_o in the numerator since this would be the voltage if the wave arrived at each element of the array at the same time. In this case we would call nV_o the maximum voltage, V_m.

The overall phase of the resulting sum is simply,

$$\phi=\frac{n\delta}{2}$$

Since δ = kdsinθ = 2πdsinθ/λ, the total voltage can be written as a function of the angle, θ,

$$V(\theta)=V_m\left[\frac{\sin\left(\frac{n\pi d\sin\theta}{\lambda}\right)}{n\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right]$$

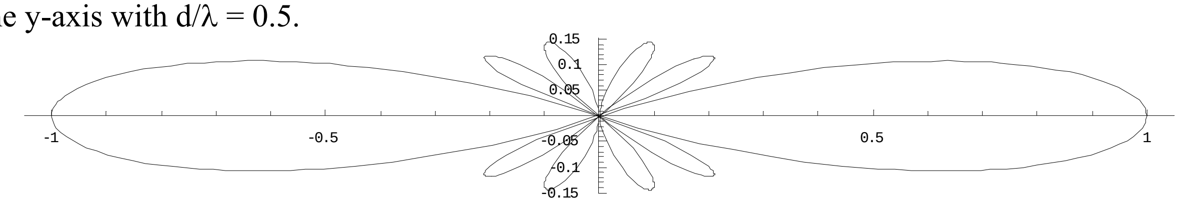
The power seen by the beamformer is then,

$$P(\theta)=\frac{V_m^2}{R}\left[\frac{\sin\left(\frac{n\pi d\sin\theta}{\lambda}\right)}{n\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right]^2$$

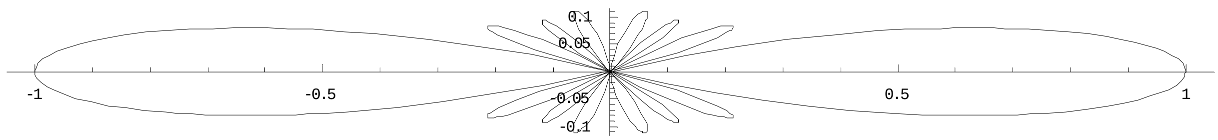
Finally, the beam pattern function is defined,

$$b(\theta)=\frac{\langle P(\theta)\rangle}{\langle P(\theta=0)\rangle}=\left[\frac{\sin\left(\frac{n\pi d}{\lambda}\sin\theta\right)}{n\sin\left(\frac{\pi d}{\lambda}\sin\theta\right)}\right]^2$$

Side lobes and maximums are dependent on the number of elements in the array. For six elements, a null can be created from a hexagon of the 6 representative phasors. This corresponds to a phase angle, δ, of 60 degrees between phasors. Additional nulls can be found when δ is 120° (triangle), 180°, 240°, and 300°. Below is the beam pattern ($\sqrt{b(\theta)}$)for a six element array along the y-axis with d/λ = 0.5.



In general, the greater the greater the number of elements, the more nulls and therefore more side lobes are created. Each lobe is narrower resulting in increased bearing resolution. Below is the beam pattern for an eight element array along the y-axis with d/λ= 0.5. Can you describe the phasor diagram that creates each of the nulls?



Directivity Index

Calculating the Directivity index for an n-element array is fairly difficult. Using the definition of Directivity Index,

$$DI = 10\log \frac{1}{\int_0^{\pi/2} b(\theta)\cos\theta d\theta}$$

we state without proof that if the beam pattern function for an n-element array is evaluated, the result is:

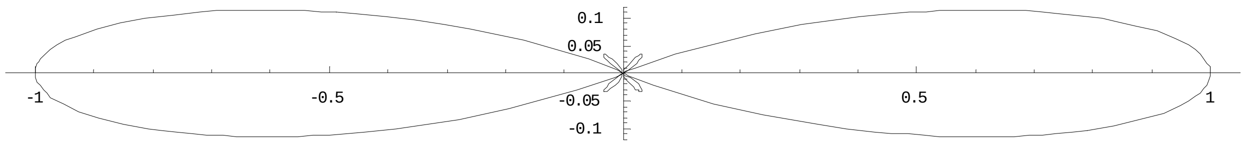
$$DI = 10\log\left[\frac{n}{1+\frac{2}{n}\sum_{\rho=1}^{n-1}\frac{(n-\rho)\sin\left(\frac{2\pi\rho d}{\lambda}\right)}{\frac{2\pi\rho d}{\lambda}}}\right]$$

Linear Arrays

A linear array is a continuous collection of many very small elements. The phasor diagram is similar to the one above with n a very large number and each individual element having a very small length. Because of this, the same beam pattern function can be used as the n-element array with the substitution that array length L = nd. Additionally, with many small elements, the denominator is the sine of a very small angle allowing us to use the small angle approximation, $\sin\alpha = \alpha$.

$$b(\theta) = \left[\frac{\sin\left(\frac{n\pi d}{\lambda}\sin\theta\right)}{n\sin\left(\frac{\pi d}{\lambda}\sin\theta\right)} \right]^2 = \left[\frac{\sin\left(\frac{\pi L}{\lambda}\sin\theta\right)}{n\frac{\pi d}{\lambda}\sin\theta} \right]^2 = \left[\frac{\sin\left(\frac{\pi L}{\lambda}\sin\theta\right)}{\frac{\pi L}{\lambda}\sin\theta} \right]^2$$

Below is the beam pattern function for a linear array along the y axis with $L/\lambda = 2$.



Nulls and Side Lobes

Nulls occur when $\sin\left(\frac{\pi L \sin\theta}{\lambda}\right) = 0$. The sine function has zeros at integer multiples of 180 degrees or π radians.

$$\frac{\pi L \sin\theta}{\lambda} = n\pi, \quad n = 1, 2, 3, \ldots$$

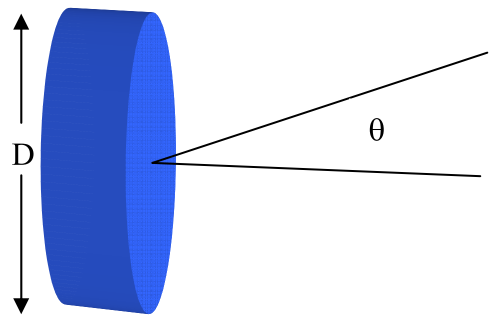
Between these nulls are secondary maxima or side-lobes that occur when the function $\frac{\sin\alpha}{\alpha}$ is a maxima. ($\alpha = \frac{\pi L \sin\theta}{\lambda}$). We can find cases where this occurs with a computer and observe that smallest value is $\alpha = 1.43\pi$. For this value, $b(\theta) = 0.04719$ and $10\log b(\theta) = -13.3\,\text{dB}$. This means that the first side lobe next to the main lobe at $\theta = 0$ degrees is reduced in amplitude by 13.3 dB.

Directivity Index

Again without proof, the directivity index of a linear array reduces to the following simple result so long as the array length is much greater than the wavelength.

$$DI = 10\log\frac{1}{\int_0^{\pi/2} b(\theta)\cos\theta d\theta} = 10\log\left(\frac{2L}{\lambda}\right)$$

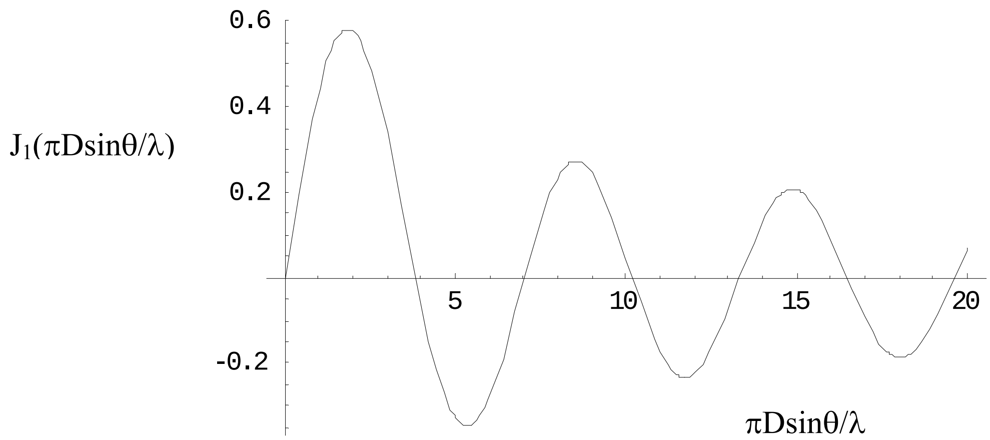
Piston Arrays



A plane piston array as shown above is thought of as composed of a very large number of elements arranged in 2 dimensions on it's surface. Since there is no fixed phase relationship between these elements, phasor addition will not work. Instead, it is necessary to integrate over the elements making up the surface. Experience has shown this is best done in polar coordinates and the results will not be repeated here. The resulting beam pattern function is

$$b(\theta) = \left[\frac{2J_1\left(\frac{\pi D \sin\theta}{\lambda}\right)}{\frac{\pi D \sin\theta}{\lambda}} \right]^2$$

where J_1 is the Bessel Function of the first order and first kind. It's values are well tabulated in mathematical handbooks much like the trigonometric functions. As seen below, maximum values and zero crossings for this Bessel function are not as orderly as the trigonometric functions.



$J_1(\pi D\sin\theta/\lambda)$ has zero crossings (nulls) at $\pi D\sin\theta/\lambda$ = 3.83, 7.02, 10.17, 13.32, 16.47, ...
$J_1(\pi D\sin\theta/\lambda)$ has extremes (near the side lobes) at $\pi D\sin\theta/\lambda$ = 1.84, 5.33, 8.54, 11.71, 14.86, ...

From this we see that the first zero crossing corresponding to a null in the beam pattern function occurs when

$$\sin\theta = \frac{3.83\lambda}{\pi D} = 1.22\frac{\lambda}{D}$$

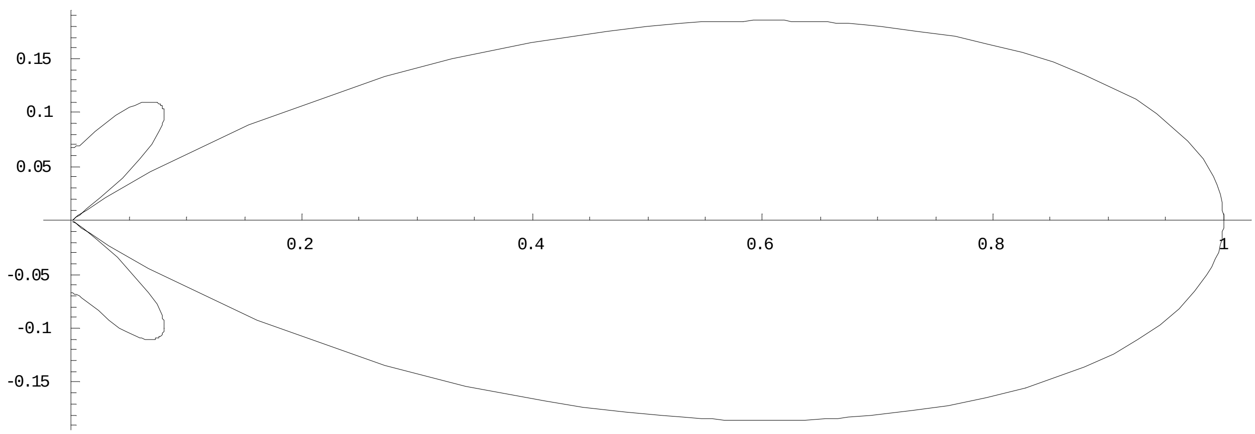
The first side lobe occurs when

$$\frac{2J_1\left(\frac{\pi D\sin\theta}{\lambda}\right)}{\frac{\pi D\sin\theta}{\lambda}} = \max$$

The actual value of the maximum corresponding to the first side lobe is found by iterating with a computer. It is near the place where $\frac{\pi D\sin\theta}{\lambda} \approx 5.33$, and the exact value is $\sin\theta = \frac{1.66\lambda}{D}$.

Note that the center beam occurred at θ = 0 where both the numerator and denominator are approaching zero.

Below is the beam pattern ($\sqrt{b(\theta)}$)for a piston array along the y-axis with D/λ = 2.0.



A Table showing the piston array results for lobes, nulls, and beam widths as well as those for linear and two element arrays appears on the following page.

	2-element array	**continuous line array**	**circular piston**
defining parameters	element separation distance – d	array length – L	array diameter - D
beam pattern function $b(\theta) =$	$\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right)$	$\left(\frac{\sin\left[\frac{\pi L}{\lambda}\sin\theta\right]}{\frac{\pi L}{\lambda}\sin\theta}\right)^2$	$\left[\frac{2J_1\left(\frac{\pi D}{\lambda}\sin\theta\right)}{\frac{\pi D}{\lambda}\sin\theta}\right]^2$
directivity index DI	$10\log\left[\frac{2}{1+\left(\frac{\sin\left(2\pi d/\lambda\right)}{2\pi d/\lambda}\right)}\right]$	$10\log\frac{2L}{\lambda}$ for $L >> \lambda$	$10\log\left(\frac{\pi D}{\lambda}\right)^2$ for $D >> \lambda$
null angles $b(\theta) = 0$ θ_{null}	$\sin\theta = (m)\frac{\lambda}{2d}$ m = 1, 3, 5, ...	$\sin\theta = (m)\frac{\lambda}{L}$ m = 1, 2, 3, ...	$\sin\theta = (z)\frac{\lambda}{D}$ z = 1.22, 2.23, 3.24, 4.24, . roots of $J_1\left(\frac{\pi D}{\lambda}\sin\theta\right) = 0$
side lobes $b(\theta)=1$ θ_{max}	$\sin\theta = m\frac{\lambda}{d}$ m = 0, 1, 2, 3	$\tan\left(\frac{\pi L\sin\theta}{\lambda}\right) = \left(\frac{\pi L\sin\theta}{\lambda}\right)$ $\sin\theta = y\left(\frac{\lambda}{L}\right)$ where y = 1.43, 2.46, 3.47, 4.4	$\sin\theta = w\frac{\lambda}{D}$ where w = 1.64, 2.68, 3.70,
half power angles $b(\theta)=0.5$ θ_{hp} $\theta_{BW}=2\theta_{hp}$ (only for beam about array axis)	$\sin\theta_{hp} = \frac{n\lambda}{4d}$ $n = 1,3,5,7,\ldots$	$\sin\theta_{hp} = 0.442\frac{\lambda}{L}$	$\sin\theta_{hp} = 0.51\frac{\lambda}{D}$

Problems:

1. Given a 2 element array with a 1.0 m spacing between elements, determine the Directivity Index assuming the frequency is 3000 Hz and c = 1500 Hz.

2. Find the directivity Index of a line of 6 elements spaced 10 cm apart when receiving sound of wavelength 30 cm.

3. The Directivity Index of a sonar array depends on all of the following except:

   a) the physical dimensions of the array.
   b) the speed of sound in the water.
   c) the layout of the hydrophones in the array.
   d) the efficiency of the array.

4. Determine the null angles from 0 to 90° of a 0.25 m active linear array operating at 25 kHz.

5. A 200m linear array is used for receiving a 300 Hz signal. What is the directivity index.

6. A continuous line array of length 150 cm is receiving sound of 5 kHz. The sound speed is 1500 m/s.
   a) Find the angles at which there is a null in the directivity pattern.
   b) Find the angles to the maximum points of all side lobes.
   c) Calculate the half power beam width.
   d) Calculate $b(\theta)$ for $\theta = 10^o, 20^o, 30^o, 40^o, 50^o, 60^o, 70^o, 80^o, 90^o$
   e) Calculate the Directivity Index.

7. Find the directivity index for a linear array of length 125 cm, when operating at 15 kHz in water where c = 1500 m/s.

8. Find the directivity index for a circular piston array of diameter 125 cm, when operating at 15 kHz in water where c = 1500 m/s.

9. A plane circular piston array of diameter 100 cm is receiving sound of frequency 7 kHz. The sound speed is 1500 m/s.
   a) Find the angles at which there are nulls in the directivity pattern
   b) Find the angles to the maximum points of all side lobes.
   c) Calculate the half-power beam width.

10. a) Design a plane circular array with a half-power beam width of 25^o at 15 kHz. The diameter of the array is _________.
    b). Design a continuous line array with a half-power beam width of 25^o at 15 kHz. The length of the array is _________.

11. What is the spacing, d, required for a 4-element line array (detecting frequencies of 10 kHz in water) so that:

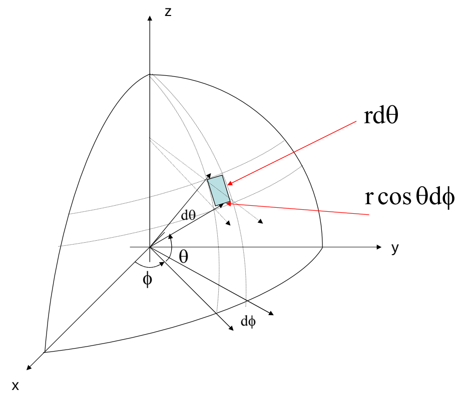
a) The first null in the beam pattern occurs at 90^o.
b) The second major lobe (of magnitude unity) in the beam pattern occurs at 90^o.
c) Compute DI for a)
d) Compute DI for b)

Directivity Index

$$DI = 10\log\frac{N_{\text{omni-directional noise}}}{N_{\text{directional noise}}}$$

$$N = \iint I_i b(\theta,\phi)\,dA$$

$$\text{but } dA = r^2\cos\theta\, d\theta\, d\phi$$



Directional Case

$$N_{ND} = 4\pi r^2 I_i$$

$$N_D = \iint I_i b(\theta,\phi)\,dA$$

$$N_D = \iint I_i b(\theta,\phi)\, r^2\cos\theta\, d\theta\, d\phi$$

$$N_D = I_i r^2 \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta,\phi)\cos\theta\, d\theta\, d\phi$$

With Rotational Symmetry

$$N_D = I_i r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta)\cos\theta\, d\theta \int_0^{2\pi} d\phi$$

$$N_D = 2\pi I_i r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta)\cos\theta\, d\theta$$

$$N_D = 4\pi I_i r^2 \int_0^{\pi/2} b(\theta)\cos\theta\, d\theta$$

DI with Rotational Symmetry

$$DI = 10\log\frac{N_{ND}}{N_D}$$

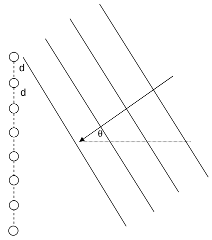
$$DI = 10\log\frac{4\pi r^2 I_i}{4\pi I_i r^2 \int_0^{\pi/2} b(\theta)\cos\theta\, d\theta}$$

$$DI = 10\log\frac{1}{\int_0^{\pi/2} b(\theta)\cos\theta\, d\theta}$$

DI for the Two-element Array

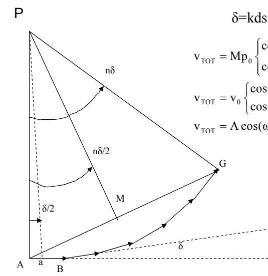
$$DI = 10\log\frac{2}{\left[1 + \frac{\sin\left(2\pi d/\lambda\right)}{2\pi d/\lambda}\right]}$$

N Element Array



$$v_1 = Mp_1(t) = Mp_{max}\cos\left(k(0) + \omega t\right)$$
$$v_2 = Mp_2(t) = Mp_{max}\cos\left(k(-\Delta x) + \omega t\right)$$
$$v_3 = Mp_3(t) = Mp_{max}\cos\left(k(-2\Delta x) + \omega t\right)$$
$$v_4 = Mp_4(t) = Mp_{max}\cos\left(k(-3\Delta x) + \omega t\right)$$
$$\vdots$$
$$v_N = Mp_N(t) = Mp_{max}\cos\left(k(-(N-1)\Delta x) + \omega t\right)$$
$$\text{output} \propto (v_1 + v_2)^2 = \left\{Mp_{max}\left[\cos(\omega t) + \cos(-\delta + \omega t) + \cos(-2\delta + \omega t) + \cos(-3\delta + \omega t) + \ldots + \cos(-(N-1)\delta + \omega t)\right]\right\}^2$$
$$\text{where } \delta = k\Delta x = kd\sin\theta$$

Phasor Addition



$$\delta = kd\sin\theta$$

$$v_{TOT} = Mp_0\left\{\begin{matrix}\cos(k[0] + \omega t) + \cos(k[-d\sin\theta] + \omega t) + \\ \cos(k[-2d\sin\theta] + \omega t) + \ldots\cos(k[-(n-1)d\sin\theta] + \omega t)\end{matrix}\right\}$$

$$v_{TOT} = v_0\left\{\begin{matrix}\cos(\omega t) + \cos(\omega t - \delta) + \cos(\omega t - 2\delta) + \ldots \\ \cos(\omega t - (n-1)\delta)\end{matrix}\right\}$$

$$v_{TOT} = A\cos(\omega t + \phi)$$

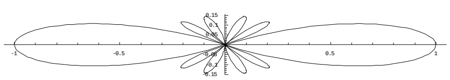
$$\frac{V(\theta)}{2} = R\sin\left(\frac{n\delta}{2}\right) \quad \text{Where R is distance AP}$$

$$\Rightarrow V(\theta) = nV_0\left[\frac{\sin n\frac{\delta}{2}}{n\sin\frac{\delta}{2}}\right]$$

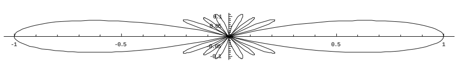
$$\Rightarrow b(\theta) = \left[\frac{\sin\left(\frac{n\pi d}{\lambda}\sin\theta\right)}{n\sin\left(\frac{\pi d}{\lambda}\sin\theta\right)}\right]^2$$

Beam Patterns for 6 and 8 Element Arrays ($\lambda/d = 0.5$)

$$\sqrt{b(\theta)}$$



6 element array oriented vertically



8 element array oriented vertically

Directivity Index for an n-Element Array

$$DI = 10\log\left[\frac{n}{1 + \frac{2}{n}\sum_{\rho=1}^{n-1}\frac{(n-\rho)\sin\left(\frac{2\pi\rho d}{\lambda}\right)}{\frac{2\pi\rho d}{\lambda}}}\right]$$

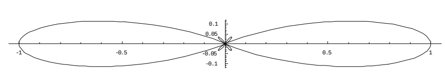
Linear Array

$$b(\theta) = \left[\frac{\sin\left(\frac{n\pi d}{\lambda}\sin\theta\right)}{n\sin\left(\frac{\pi d}{\lambda}\sin\theta\right)}\right]^2 = \left[\frac{\sin\left(\frac{\pi L}{\lambda}\sin\theta\right)}{n\frac{\pi d}{\lambda}\sin\theta}\right]^2 = \left[\frac{\sin\left(\frac{\pi L}{\lambda}\sin\theta\right)}{\frac{\pi L}{\lambda}\sin\theta}\right]^2$$

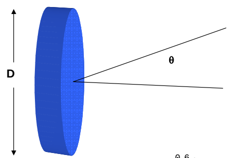
Nulls: $\frac{\pi L\sin\theta}{\lambda} = n\pi, \quad n = 1, 2, 3, \ldots$

$$DI = 10\log\frac{1}{\int_0^{\pi/2} b(\theta)\cos\theta\, d\theta} = 10\log\left(\frac{2L}{\lambda}\right)$$

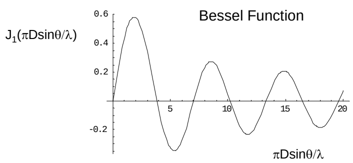
Beam Pattern for a Vertical Linear Array $L/\lambda = 2.0$



Piston Array



$$b(\theta) = \left[\frac{2J_1\left(\frac{\pi D \sin\theta}{\lambda}\right)}{\frac{\pi D \sin\theta}{\lambda}}\right]^2$$

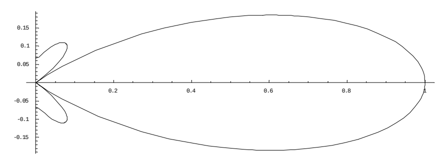


$J_1(\pi D\sin\theta/\lambda)$ has zero crossings (nulls) at $\pi D\sin\theta/\lambda$ = 3.83, 7.02, 10.17, 13.32, 16.47, ...
$J_1(\pi D\sin\theta/\lambda)$ has extremes (side lobes) at $\pi D\sin\theta/\lambda$ = 1.84, 5.33, 8.54, 11.71, 14.86, ...

Beam Pattern for a Piston Array

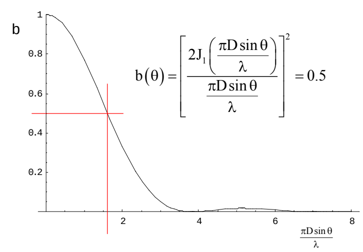
$D/\lambda = 2.0$

$\sqrt{b(\theta)}$



	2-element array	continuous line array	circular piston
defining parameters	element separation distance – d	array length – L	array diameter - D
beam pattern function b(θ) =	$\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right)$	$\left(\frac{\sin\left[\frac{\pi L}{\lambda}\sin\theta\right]}{\frac{\pi L}{\lambda}\sin\theta}\right)^2$	$\left[\frac{2J_1\left(\frac{\pi D}{\lambda}\sin\theta\right)}{\frac{\pi D}{\lambda}\sin\theta}\right]^2$
directivity index DI	$10\log\left[\frac{2}{1+\left(\frac{\sin(2\pi d/\lambda)}{2\pi d/\lambda}\right)}\right]$	$10\log\frac{2L}{\lambda}$ for $L >> \lambda$	$10\log\left(\frac{\pi D}{\lambda}\right)^2$ for $D >> \lambda$
null angles b(θ) = 0 θ_{null}	$\sin\theta = (m)\frac{\lambda}{2d}$ $m = 1, 3, 5, \ldots$	$\sin\theta = (m)\frac{\lambda}{L}$ $m = 1, 2, 3, \ldots$	$\sin\theta = (z)\frac{\lambda}{D}$ $z = 1.22,\ 2.23,\ 3.24, 4.24, \ldots$ roots of $J_1\left(\frac{\pi D}{\lambda}\sin\theta\right) = 0$
side lobes b(θ)=1 θ_{max}	$\sin\theta = m\frac{\lambda}{d}$ $m = 0, 1, 2, 3 \ldots.$	$\tan\left(\frac{\pi L\sin\theta}{\lambda}\right) = \left(\frac{\pi L\sin\theta}{\lambda}\right)$ $\sin\theta = y\left(\frac{\lambda}{L}\right)$ where $y = 1.43, 2.46, 3.47, 4.48, \ldots$	$\sin\theta = w\frac{\lambda}{D}$ where $w = 1.64, 2.68, 3.70, \ldots.$
half power angles b(θ)=0.5 θ_{hp} $\theta_{BW}=2\theta_{hp}$ (only for beam about array axis)	$\sin\theta_{hp} = \frac{n\lambda}{4d}$ $n = 1, 3, 5, 7, \ldots$	$\sin\theta_{hp} = 0.442\frac{\lambda}{L}$	$\sin\theta_{hp} = 0.51\frac{\lambda}{D}$

Beam width of a piston array



$$b(\theta) = \left[\frac{2J_1\left(\frac{\pi D \sin\theta}{\lambda}\right)}{\frac{\pi D \sin\theta}{\lambda}}\right]^2 = 0.5$$

$$\frac{\pi D \sin\theta}{\lambda} = 1.6 \qquad \sin\theta = \frac{1.6}{\pi}\frac{\lambda}{D} = .51\frac{\lambda}{D}$$