Doppler Distortion

Piston Vibrating in a Tube Including the Effect of Excursion

In other sections of this web site the wave produced by a vibrating piston is analyzed, but the effect of the piston excursion is neglected. That is, the piston has velocity, but the piston face is always treated as if its deviation from a fixed location is negligible. Here we obtain an exact solution where the excursion is taken into account. The excursion introduces harmonic distortion even when the piston is vibrating with a pure sinusoidal motion at a single frequency. When two frequencies are present intermodulation frequencies are created. Calling this "Doppler Distortion" may be a misnomer, but that is what people have called it for a long time.

There are people who believe that this is all a myth. As a believer in physics and mathematics, for me the derivation below is conclusive, and ends the debate. For the mathematically challenged among us, there is a heuristic description of the cause of this distortion at the end of this section.

Introduction: Fundamental Cause of Doppler Distortion

Consider a vibrating piston whose position x_p is defined by the equation

$$(1) \quad x_p(t) = -\frac{v}{\omega}\cos\omega t$$

The piston velocity is

$$(2) \quad v_p(t) = v\sin\omega t$$

Now consider an ideal sound wave in the tube, a single harmonic tone at the same frequency ω. We assume that the sound obeys the linear one-dimensional wave equation, i.e. that the sound intensity is within the region where air behaves linearly. Neglecting a multiplicative constant, the mean molecular velocity u, in the x-direction, at any point $x \geq x_p$, and at any time t satisfies

$$(3) \quad u(x,t) = \sin\left(\omega\left[t - \frac{x}{c}\right]\right)$$

Where c is the speed of sound. This form follows directly from the wave equations.

The crux of the situation is this:

At any fixed point x in the tube, the time variation of u is a pure sine wave. The time variation of the piston velocity is a pure sine wave when it is measured at the face of the piston, $x=x_p$, The time variation of u is not a pure sine wave at $x=x_p$. Thus the piston velocity of equation (2) does not match the sound wave velocity of equation (3) at the interface of the piston and the air. Since this is the basic boundary condition that must be satisfied, equation (3) is not a valid solution for the assumed piston motion. So at this point we know the piston is not generating a pure tone - i.e. there is Doppler distortion.

This velocity mismatch raises several interesting issues. We will first determine what tones are generated by a piston velocity defined by equation (2). We will return to a discussion of the interesting issues at the end.

Harmonic Vibration at a Single Frequency: Exact Solution

The necessary and sufficient conditions for an exact solution for the sound wave are: (1) the solution satisfies the wave equation, and (2) the wave velocity equals the piston velocity at the piston face. It should be emphasized that this means an exact solution everywhere in front of the piston, from smack dab on the piston face to plus infinity.

As a trial solution, consider

$$(4) \quad u(x,t) = b_0 + \sum_{n=1}^{N} a_n \sin\left(n\,\omega\left[t - \frac{x}{c}\right]\right) + b_n \cos\left(n\,\omega\left[t - \frac{x}{c}\right]\right)$$

By construction this trial solution always exactly satisfies the wave equation. Equation (4) defines the wave velocity for any value of x, at any time t. As discussed above, we need to match the piston velocity at the location of the piston face at any given moment. Therefore we will have a valid solution if we can find coefficients such that at $x=x_p(t)$ the wave velocity matches the piston velocity, meaning

$$(5) \quad v \sin\omega t = b_0 + \sum_{n=1}^{N} a_n \sin\left(n\left[\omega t + \frac{v}{c}\cos\omega t\right]\right) + b_n \cos\left(n\left[\omega t + \frac{v}{c}\cos\omega t\right]\right)$$

Equations (4) and (5) have some superficial similarity, but the crucial difference is that equation (4) gives the velocity as a function of both position and time, whereas equation (5) gives the velocity as a function of time alone at one loci in space, namely the piston face. So equation (5) is used to solve for the coefficients, and then they can be plugged into equation (4) to define the wave everywhere in the tube. If the objective is simply to know the resulting harmonic content, that is given directly by the coefficients themselves.

Being a lazy type, I don't even want to think about solving this nasty equation analytically. But it turns out that a pretty simple Matlab program generates very accurate solutions very quickly (I will send the program to anyone who wants it).

A piston in a tube is ideally loaded, and doesn't need to move much to generate very loud sound levels. For example, at a sound level of 117 dB SPL in the tube the peak piston velocity is only .0488 meters per second, and from equation (5) the first harmonic is 83 dB down from the fundamental. But a piston in an infinite baffle has to move a lot more. A loudspeaker cone with a peak excursion of 5 millimeters at 100 Hz moves at a peak velocity of 3.14 meters per second. At this velocity first harmonic from equation (5) is down 47 dB. A plot of the harmonics for a very high peak cone velocity of 10 meters per second is shown here [34 kB].

There is a constant velocity term b_o in equation (5). Much to my delight the numerical results for b_o from equation (5) agree perfectly with the constant that I predicted a long time ago, based on totally different considerations; see equation (36) in the section on plane waves. Since this is a pretty interesting result by itself, I have appended further discussion of this effect at the end of this section.

In any case, even with a "perfect" piston vibration, and with the air behaving in a perfectly linear manner, there are harmonics. The non-linearity arises strictly from the effect of the piston excursion.

Harmonic Vibration at Two Frequencies: Exact Solution

The numerical solution is in principle easily extended to the two-frequency case. The second frequency term is simply added to equations (1) and (2). To formulate equation (5), the "x" in equation (4), where we need to match the velocity, becomes the new $x_p(t)$ for the combined frequencies. Additional terms are added to equation (5) at frequencies corresponding to the new harmonics and intermodulation frequencies. Note that you can put whatever potential frequency terms you want in equation (5), as long as they satisfy the wave equation. If the frequency is not present, the program simply outputs a zero for that coefficient. In fact many of the coefficients output by the program are totally negligible. The "proof of the pudding" is if coefficients can be found such that the new equation (5) equals the new combined velocity. With the proliferation of terms the numerical solution is a little trickier than the one-frequency case. My current Matlab program only works for a large ratio of the high frequency to the low frequency. But for this situation, and with a little care, accurate solutions can still be found.

As an example, I have evaluated the case of frequencies of f1=50 and f2=1000 Hz, with peak velocities of 1 and .1 meters per second respectively. A plot of the piston velocity overlaid with the wave velocity is shown here [35 kB]. In other words, this is a plot of both the right and left hand sides of equation (5). The first harmonic of f1 is down 56.7 dB relative to the fundamental at f1. The first harmonic of f2 is down 76.7 dB relative to the fundamental at f2. The first sidebands at f2-f1 and f2+f1 are down 30.7 dB relative to the fundamental at f2. Therefore the intermodulation distortion is the dominant effect when these two frequencies are present. A plot of the harmonic content near f2 is shown here. [33 kB].

Harmonic Vibration at a Single Frequency: Approximate Solution

The above analysis is not easy to grasp intuitively. There is an approximate analysis that yields an answer quite close to the exact solution, and is much more intuitive. Given a piston vibrating at a single frequency, imagine a probe at some fixed location in the tube that measures the sound signal. To keep the argument simple, ignore any harmonics at this point, and assume we have a pure sine wave with wavelength λ . If the center of vibration of the piston is moved forward a distance d to a new fixed location, the phase of the signal measured by the probe will increase by a fixed value of $2\pi d/\lambda$ radians. So a forward motion imposed on the piston changes the phase. If we assume that the same phase change occurs when the piston face is displaced dynamically from its center of vibration, then there is a phase modulation of the signal due to the piston excursion. Assuming such a phase modulation, for the single frequency case with the piston location given by equation (1) the resulting signal measured by the probe, neglecting unimportant constants, would be

$$(6) \quad s(t) = \sin\left(\omega\left[t + \frac{x_p(t)}{c}\right]\right) = \sin(\omega t - \frac{v}{c}\cos\omega t)$$

Equation (6) can be expanded in an infinite series of harmonics of the fundamental

$$(7) \quad s(t) = -J_1\left(\frac{v}{c}\right) + \left[J_0\left(\frac{v}{c}\right) + J_2\left(\frac{v}{c}\right)\right]\sin\omega t - \left[J_1\left(\frac{v}{c}\right) - J_3\left(\frac{v}{c}\right)\right]\cos 2\omega t + \cdots$$

Where J_n is the Bessel function of the 1st kind. For a quite high piston velocity of 20 meters per second, which produces a first harmonic 30.7 dB below the fundamental, the coefficients produced by equation (7) agree with the three equivalent coefficients produced by equation (5) to within 0.2%. The coefficient of the 2^{nd} harmonic term is approximately equal to v/(2c) for v<<c.

Harmonic Vibration at Two Frequencies: Approximate Solution

Again to keep things relatively simple, assume that the piston excursion is dominated by the movement at a low frequency ω_1. Then the high frequency signal at ω_2 will have an approximate phase modulation given by

$$(8) \quad s(t) = \sin\left(\omega_2\left[t + \frac{x_p(t)}{c}\right]\right) = \sin(\omega_2 t - \frac{\omega_2 v}{\omega_1 c}\cos\omega t)$$

So the result is similar, except the effect is magnified by the ratio of the frequencies. The approximate coefficient solution for the 2nd harmonic, $(\omega_2 v)/(2\omega_1 c)$ also gives a result quite close to the two-frequency case presented above.

Interesting Issues

The approximate analysis is so accurate one might ask why I went to the trouble of generating the exact solution. I would turn this around, and say that what interests me is why the approximate solution is not exact! For very high velocities there is a significant disagreement. So the dynamic phase shift argument is approximate and needs verification, which is provided by the exact analysis.

The common method of computing Doppler distortion for the two frequency case is to: (1) compute a dynamic Doppler frequency offset based on the cone velocity at the low frequency, and (2) apply that as a FM modulation of the higher frequency. It is not obvious at first sight, but it is not difficult to show that (with one minor qualification) this is mathematically identical to applying a phase modulation based on the cone position, as done here. That is, the sidebands predicted by both analyses have exactly the same magnitude and phase. But for the single frequency case there isn't a Doppler frequency shift per se, and the phase modulation approach is the only way to get an answer.

Another question is: if an amplifier outputs a perfect sine wave voltage, what is the cone motion of a good speaker? I went back over the details of the Thiele-Small analysis to confirm that it is in fact the sinusoidal motion assumed in equations (1) and (2), as long as the speaker is in its linear region. Mainly this means that the voice coil stays in the linear region of the magnetic field, and the cone excursion is in the linear region of the suspension.

An important question one might ask is if there are similar effects during a sound recording process, and maybe the "distortion" actually washes out. If the musical instrument is a piston in the tube, and if the microphone is a piston in the tube, and if the speaker is a piston in the tube, in principle I think this is possible. But there is still an immediate qualification: the harmonics are amplitude dependent, so if, for example, the playback volume is different than the original "music" volume, the harmonics in the playback will be different - i.e. distorted. In the real world of course the sound source, microphone, and speaker all have very different velocities of motion.

What about Fourier analysis? Certainly the piston motion can be represented by a Fourier transform, and all of the usual spectral analyses can be performed. The same is true for the sound wave velocity generated by the piston motion. However, the two spectra will differ. In most practical cases the difference is small, but the sidebands produced by the modulation of a high tone by a low tone can certainly be significant.

The last question regards beat notes. Since my Matlab program only works for widely separated frequencies, I can't look for these, but you certainly have to guess that they would be there as well.

This type of distortion can only be reduced by minimizing the motion of the surface generating the sound. A larger surface moves less for a given sound level. The large diaphragm of an electrostatic speaker moves very little, and this might at least partially explain the wonderful clarity of some ES speakers. A three-way system is better than a two-way in this regard. A 4th order crossover is better than a 1st or 2nd order. Of course there are other factors involved in all of these design choices, and Doppler distortion may not be the most critical factor.

Heuristic Description of Doppler Distortion

Consider a piston vibrating in a perfect sinusoidal motion at 500 Hz. The wave generated in the tube will look a lot like a 500 Hz sine wave at first glance. But let's consider the zero crossings of the wave. Say the piston position vibrates between x=0 and x=1 centimeter (cm), and begins at x=0 with zero velocity at t=0. So the velocity wave begins with a pulse shape that starts at zero followed by a rising pulse slope. The pulse begins to travel down the tube at the speed of sound, 344 meters per second. At t=1 millisecond (ms) the piston is at x=1 cm and its velocity is zero. At this instant in time the wave consists of a positive half-cycle pulse with the first zero that has traveled down the tube to x=34.4 cm, and the second zero at x=1 cm. The zeros are 33.4 cm apart, and they will remain at that separation as the pulse continues to travel down the tube. At t=2ms the piston is back at x=0 and again has zero velocity. At this instant in time the wave consists of one full cycle, a positive pulse followed by a negative pulse, with zeros that are now at x=68.8 cm, x=35.4 cm, and x=0 cm. This pattern with a wavelength of 68.6 cm will then continue to be replicated as time goes on. The crucial point is that the zero crossings are not equally spaced; on the positive pulse section they are 33.4 cm apart, and on the negative pulse section they are 35.4 cm apart. We have a distorted sine wave.

Mass Transport for a Piston Vibrating in a Tube

To understand this, it is necessary to understand a sound wave as the motion of air molecules. For a pure tone, defined by equation (3), the average molecular velocity forms a perfect sine wave. At the wave peaks, the molecules, on average, are moving forward. At the troughs, they are moving backward at exactly the mirror image of the forward velocity. At the wave crests the molecular density is higher than at the wave troughs. Therefore there is a net flow of molecules in the direction of wave propagation. If you calculate the power flow in watts per square meter, you find that the wave is carrying more power than the piston is putting out. (In this calculation piston excursion is neglected). In other words, you are literally creating energy "out of thin air." Now everyone knows that there really isn't a net flow of molecules, and you can't violate the conservation of energy. What's wrong is that equation (3) by itself is not the right solution. If you add the constant term mentioned above, both problems are perfectly cleared up: there is no net flow of molecules, and the piston puts out exactly as much wattage as the wave carries down the tube. All of the physics and math to back all of this up is given elsewhere on this web site. Follow the link above, and subsequent links.

Acknowledgement: I benefited from technical exchanges in carrying out this analysis. Due to issues beyond my control that is all I am going to say.

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