The Smith Chart

Contents

Introduction

This article deals with ideal transmission lines for electrical waves. If you would like a review of sinusoidal signals, phasors and transmission line equations, please read Backward Waves. We shall use the same notation here, except that the coordinate z that specifies location along a transmission line has z = 0 at the load instead of the source, and that we also use z as the normalized impedance Z/Z_o, where Z_o is the characteristic impedance of the line. To avoid confusion, -z is often represented by d.

Our signals are sinusoidal waves of frequency f and wavelength λ, and fλ = v, the velocity of the waves. In free space, v = c = 2.9978 x 10^8 m/s, approximately. In any material medium, v = c/n, where n is the index of refraction of the medium. Velocity on a transmission line is usually expressed by the *velocity factor* 1/n instead. When calculating wavelength on a transmission line, the velocity factor must be taken into account. Indeed, λ = c/nf = velocity factor x c/f. For theoretical work, the angular frequency ω = 2πf, and the propagation constant k = 2π/λ are more convenient.

For clarity, we shall consider only ideal lines in this article, those with no series resistance or shunt conductance. Actual lines approximate ideal lines rather closely, so this is not a serious limitation. Energy is conserved on an ideal line; the power out is the power in. The principal parameters of a line are its capacitance C and inductance L per unit length. The wave velocity is then v = 1/√(LC) and the characteristic impedance is Z_o = √(L/C). These two parameters are generally quoted for any transmission line material.

Coaxial cable RG-8/U has Z_o = 53Ω and 1/n = 0.66. This velocity factor is typical of polyethyene (PE) insulation. RG-59/U, with Z_o = 73Ω, has the same insulation and velocity factor. RG-141/U, with polytetrafluoroetylene (PTFE) insulation, has Z_o = 50Ω and 1/n = 0.70. PE foam is mainly air, so RG-8/U(foam) has a velocity factor of 0.80 but about the same Z_o = 50Ω. Coaxial cable has the great advantage that the fields are totally enclosed. The molded 300Ω "twin-lead" has a velocity factor of 0.82. Parallel-wire lines in air have even larger velocity factors, usually about 0.95.

A parallel-wire line with conductors of diameter d spaced a distance s betweeen centerlines has C = $\pi\varepsilon/\cosh^{-1}(s/d)$ and L = $(\mu/\pi)\cosh^{-1}(s/d)$. The product LC = με = $1/c^2$, so the ideal velocity factor is 1. The characteristic impedance is $Z_o = (\cosh^{-1}/\pi)\sqrt{(\mu/\varepsilon)}$. $\sqrt{(\mu/\varepsilon)}$ = 377Ω, the wave impedance of free space. For d = 2mm and s = 20mm, Z_o = 359Ω. The inverse hyperbolic cosines are calculated directly by the HP-48G, but can also be expressed in terms of natural logarithms.

Waves on Ideal Transmission Lines

On an ideal transmission line, there is generally a wave moving from source to load $V'e^{-jkz}$, and a wave moving from load to source $V''e^{jkz}$. The coefficients V' and V" are constants independent of z, since the line is lossless. The corresponding currents are $(V'/Z_o)e^{-jkz}$ and $-(V''/Z_o)e^{jkz}$. All quantities are multiplied by the time factor $e^{j\omega t}$, which is understood. The total voltage and current at any point are the sums of the contributions of the two waves.

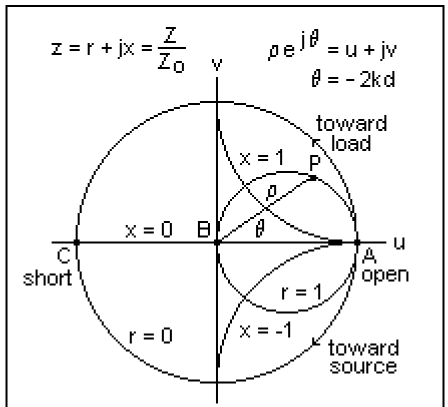
If the line is terminated at z = 0 with an impedance Z_L, then this must be the ratio of the total voltage to the total current at that point, or $Z_L = Z_o(V' + V'')/(V' - V'')$. This condition establishes the ratio of V" to V'. Let us define the *reflection coefficient* $\boldsymbol{\rho}(z)$ as the complex ratio $(V''e^{jkz})/(V'e^{-jkz}) = (V''/V')e^{2jkz}$. We then have $z(0) = Z_L/Z_o = [1 + \boldsymbol{\rho}(0)]/[1 - \boldsymbol{\rho}(0)]$. This equation can be inverted to give $\boldsymbol{\rho}(0) = [z(0) - 1]/[z(0) + 1]$. The normalized load impedance determines the reflection coefficient at the load.

At any other point on the line, a similar equation holds, but the complex reflection coefficient varies in a simple way. In fact, $\boldsymbol{\rho}(z) = \boldsymbol{\rho}(0)e^{2jkz}$. That is, only the phase changes, while the magnitude remains constant. The magnitude will be represented by ρ. In the usual case when the termination is resistive, $\rho = [r(0) - 1]/[r(0) + 1]$. As we move a distance d towards the source, $\boldsymbol{\rho} = \rho e^{-2jkd}$. Therefore, the complex number $\boldsymbol{\rho}$, represented as a vector, rotates clockwise through an angle $(4\pi/\lambda)d$. When d is $\lambda/2$, a half-wavelength, the vector has rotated through a complete circle.

The maximum voltage on the line will be $|V'| + |V''|$ and the minimum will be $|V'| - |V''|$. The ratio of the maximum to the minmum voltage is called the *voltage standing wave ratio*, VSWR, and there is an analogous definition for the current standing wave ratio. Clearly, maximum voltage corresponds to minimum current, and vice versa. The VSWR $S = (1 + |\rho|)/(1 - |\rho|)$ in terms of the magnitude of the reflection coefficient. If $|\rho|$ is zero, then S = 1 and the maximum voltage is constant along the line. If $|\rho| = 1$, then S is infinite, and there are points of zero voltage which correspond to points of maximum current, called *nodes*.

The Smith Chart

If we let $\boldsymbol{\rho} = u + jv$, this can be plotted in the (u,v) plane in the usual way of representing complex numbers. The normalized impedance, $z = Z/Z_o = [1 + \boldsymbol{\rho}]/[1 - \boldsymbol{\rho}]$ is a function of $\boldsymbol{\rho}$, and so its real and imaginary parts, $z = r + jx$, can be expressed in terms of the real and imaginary parts of $\boldsymbol{\rho} = u + jv$. If lines of r = constant and x = constant are drawn on the diagram, the result is called the *Smith Chart*, which is shown in the figure. This is a rather complex figure, but will repay careful study.



Smith Chart

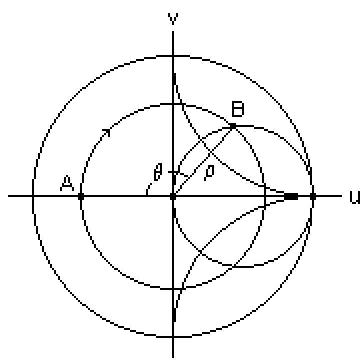
First of all, the vertical and horizontal axes are v and u, the imaginary and real parts of $\boldsymbol{\rho}$, which is normally represented in polar form. The outer circle corresponds to ρ = 1 as well as to r = 0. The equation of the circle for a resistance r is $[u - (r/r+1)]^2 + v^2 = (1/1+r)^2$, as can be seen by expressing z in terms of u and v, and finding the real part. The circle passing through points A and the origin B, of radius 1/2, corresponds to r = 1. Finally, point A corresponds to $r = \infty$.

The normalized reactance, x, is also constant on circles which pass through A and have their centres on the v-axis. These circles are easily found to be $(u - 1)^2 + (v - 2/x)^2 = (2/x)^2$. The parts of the circles for x = +1 and x = -1 are shown. Positive, or inductive, reactance is above the u-axis, while negative, or capacitive, is below. The u-axis corresponds to x = 0, or a z that is purely resistive.

We note that point C corresponds to r = 0, or to a short at the load end of the line. Point B corresponds to r = 1, so the line is terminated in its characteristic resistance. At point A, $r = \infty$, or the line is open. Either point A or point C makes ρ = 1. As we proceed toward the source, the vector $\boldsymbol{\rho}$ rotates clockwise from whichever point describes the particular termination. While r remains zero, x goes through the complete range of values from $+\infty$ to $-\infty$. This will be the reactance at any point along the line, and in particular, at the source. By changing the length of the line, we can present any desired reactive impedance to the source; at two points the reactance is zero, while the resistance is zero or infinity.

If the line is terminated as at point B, then $\rho = 0$ (and doesn't go anywhere), while the impedance presented to the source is constant at $z = 1$, or $Z = Z_0$. This important case is a *matched line*, and there is no reflection at the load end.

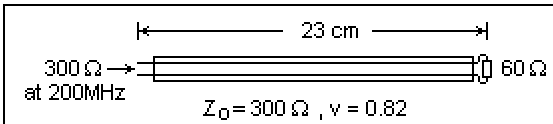
A point P at an arbitrary location is shown also. It happens to lie on the $r = 1$ circle, so if it represents the termination of a line, the termination impedance has this real part, and some reactive part as well. This point determines a reflection coefficient ρ and an angle θ which together determine the complex $\boldsymbol{\rho}$. Any other point on the line is represented by some point on a circle of radius ρ with centre at B.



Impedance Transformation

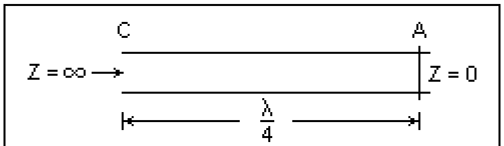
This is important enough to be represented on a separate diagram. Let us assume that a line is terminated by a resistance at point A. For concreteness, suppose $\rho = 0.6$ and $r = 0.2$ at the point A. Suppose we are dealing with a 300Ω line at 200 MHz. The termination is then 0.2 x 300 = 60Ω. Suppose B represents the source end of the line. The angle θ is 135° (say), so 135° = 2 $(2\pi/\lambda)d = 720°d/\lambda$. If the velocity coefficient is 0.82, the wavelength is 0.82 x c / 200MHz = 1.23 m. Now we can find the actual length of the line: d = (135/720)(1.23) = 23 cm. This end of the line is on the $r = 1$ circle, so the resistive component of the impedance is 300Ω. On an actual Smith Chart, we could also read off the reactance as well. Let's suppose it is $x = 2$. Then X = (2)(300) = 600Ω, inductive. Therefore, at point B, the impedance looking into the line is 300 + j600 Ω.

The physical line is shown in the figure at the right. It is represented as constructed from 300Ω plastic twinlead. The 60Ω resistor is not a standard value, but 56Ω would do about as well. Measuring the impedance at the input is a little more difficult, unless you have the very expensive insruments that can do it directly.



Impedance Transformation

Lines that are exactly a quarter-wavelength long have interesting properties. It is clear that the reflection coefficients at the two ends are simply negatives of each other, at the ends of a diameter in the Smith Chart. This means that if $Z/Z_o = (1 + \rho)/(1 - \rho)$ at one end, then $Z'/Z_o = (1 - \rho)/(1 + \rho) = Z_o/Z$ at the other, or $ZZ' = Z_o^2$. This is called a *quarter-wave transformer*. Remember that when you design such a transformer, it will work as intended only at the design frequency, for only then is it a quarter-wave long.



Quarter-Wave Stub

A quarter-wave line shorted at one end, as shown in the figure, is called a *quarter-wave stub*, and presents a very large impedance at its open end. Such stubs can be used to support a transmission line. Although there will be a DC path to ground, signal frequencies will be isolated. They will also act as pretty good bandpass filters, too, since only the signal current will not be shorted out.

In this article, I have only presented the theory of the Smith Chart with a few examples. It has been used to solve problems of many standard types that arise in transmission line design. Of course, all the calculations can be done with a pocket calculator or computer, but the chart has the great advantage of giving a *graphic* picture of conditions that can give a deeper understanding and help in solving unusual problems. The interested reader should certainly examine an actual Smith Chart to appreciate how easy it is to use, and how quickly it provides answers.

References

S. Ramo, J. R. Whinnery and T. Van Duzer, *Fields and Waves in Communication Electronics* (New York: John Wiley & Sons, 1965). pp. 31-41. Fig. 1.20b is a Smith Chart that can be copied if no other source is available.

Return to Tech Index

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